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A projection based approach to the Clebsch–Gordan multiplicity problem for compact semisimple Lie groups: I. General formalism

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Abstract. We describe a method, analogous to the Elliott method for $O(3) \subset U(3)$, for resolution of the multiplicity in the decomposition into irreducibles of the tensor product of two irreducible representations of a compact semisimple Lie group G. The method is based on a decomposition of highest weight vectors in the product representation into direct product states, focusing on components of the form $e \otimes e_+^{\mu}$ with e a weight vector and e_+^{μ} a highest weight vector. The weight of the vector e corresponds to the 'shift' weight in a tensor operator formulation of the problem. We use a result from an earlier paper, based on the fact that a tensor product representation can be generated cyclically from the product of a highest and a lowest weight vector, to give an explicit characterisation of the space of shift weight vectors e that can appear in the decomposition of a highest weight state. This characterisation is in terms of lowering operators in the complexified Lie algebra L of G, and closely parallels Verma's well known enveloping algebra characterisation of the highest weight states of finite-dimensional irreducible representations of complex semisimple Lie algebras. The result enables the problem of determining the multiplicity structure of tensor product representations to be recast as a much simpler problem of spectral analysis of weight spaces of irreducible representations under sl(2) subalgebras of L. We pursue some implications of the method for the explicit computation of Wigner coefficients, and show that the matrix elements of the relevant projection operators can be expressed as ratios of polynomial functions of known degree in the highest weight components. Two following papers give an application of the method to $U(n)$ and describe its properties in the asymptotic (classical) limit of large quantum numbers, which parallel the asymptotic properties of Elliott's scheme for the $O(3) \subset U(3)$ problem.

1. Introduction

The definition and computation of the Wigner coefficients for the unitary, orthogonal, symplectic and other compact semisimple Lie groups is well known to be a complex problem because of the multiplicity of irreducible subrepresentations occurring in the decomposition of tensor products. Although the multiplicity in representations with low quantum numbers can usually be resolved on a case by case basis by introducing additional Hermitian labelling invariants or some form of projection, in the longer term it is essential to develop more general labelling and computational procedures which exploit the structure of the group itself. General procedures of this kind have been specified by Biedenharn, Louck and co-authors (Lohe *et al* 1977 and earlier papers) for the cases of U(3) and the adjoint representations of arbitrary U(n).

In the case of the well studied embedding $O(3) \subset U(3)$, one of the simplest to exhibit a multiplicity problem, a variety of effective general labelling schemes is available, of which the earliest, due to Elliott (1958), is known to have a natural physical interpretation in the classical (large quantum number) limit. In this paper we describe a method for resolving the multiplicity in tensor product representations using an approach similar in spirit to the Elliott scheme. The multiplicity-resolved states are defined by projection from a specific set of simple product states whose behaviour under projection can be characterised straightforwardly in terms of the action of the lowering operators in the group's Lie algebra. In the Elliott scheme, projection is from a highest weight state for U(3), while in this paper we resolve tensor product multiplicities by projection from states of the form $e \otimes e_+^\mu$ where e_+^μ is a highest weight vector. The behaviour of these states under projection is amenable to detailed characterisation for the following reasons.

Let $V(\lambda)$ and $V(\mu)$ be irreducible modules carrying representations of a compact semisimple Lie group G. Let the distinct weights occurring in $V(\lambda)$ be $\lambda_1, \lambda_2, \ldots, \lambda_m$, and suppose that the numbering respects the partial order on the set of weights induced by the positive roots, i.e. $\lambda_i > \lambda_j$ iff $i > j$. λ_1 is thus the lowest weight in $V(\lambda)$ and λ_m the highest. Denote the subspace of $V(\lambda)$ of vectors of weight λ_i by $V_i(\lambda)$. It is well known that in decomposing the tensor product module $V(\lambda) \otimes V(\mu)$ the multiplicity with which the irreducible component of highest weight $\lambda_i + \mu$ occurs cannot exceed the dimension of $V_i(\lambda)$. As shown in a previous paper (Gould and Edwards 1984), this result can be greatly sharpened: for each highest weight, we can define an explicit subspace of $V_i(\lambda)$, denoted $V_{i,\mu}(\lambda)$, whose dimension is equal to the multiplicity of $\lambda_i + \mu$ in $\lambda \otimes \mu$. Furthermore, projecting highest weight vectors (HWV) of weight $\lambda_i + \mu$ contained in $V(\lambda) \otimes V(\mu)$ onto the space of simple product vectors of the form $e \otimes e_+^\mu$, where e_+^μ is a highest weight vector in $V(\mu)$, sets up a bijection between the space spanned by the HWV of weight $\lambda_i + \mu$ (whose dimension is also equal to the multiplicity of $\lambda_i + \mu$ in $\lambda \otimes \mu$) and the space $V_{i,\mu}(\lambda)$: the range of this projection is the space $V_{i,\mu}(\lambda) \otimes e_+^\mu$.

In this paper and the two following we pursue some implications of this result for defining and computing an explicit solution to the Clebsch-Gordan problem for the compact semisimple Lie groups. After setting up the notation in § 2, we recall in § 3 the definition of the space $V_{i,\mu}(\lambda)$ in terms of the generators of G, and sketch the proof of the result above, which is based on the fact that the product vector $e_+^\alpha \otimes e_-^\beta$, with e_+^α a highest weight vector for $V(\alpha)$ and e_-^β a lowest weight vector for $V(\beta)$, is cyclic under G for the representation $\alpha \otimes \beta$. In § 4 we show how to define a set of labelled Wigner coefficients between highest weight vectors of weight $\lambda_i + \mu$ and product vectors in $V_{i,\mu}(\lambda) \otimes e^\mu$ and prove that these coefficients are rational polynomial functions of the integer components of μ. We specify the programming steps necessary to compute these coefficients. General Wigner coefficients may be obtained from those defined in this paper by application of the group generators, which have known matrix elements. In the two following papers, we study the multiplicity problem in detail for the groups U(3) and U(4), establishing an approach to multiplicity resolution which will extend (albeit with substantial computational effort) to higher U(n) and to the other compact semisimple Lie groups; we then pursue the greatly simplified properties of the Wigner coefficients we define in the asymptotic limit of large quantum numbers, obtaining results which closely parallel those well known to hold for Elliott's solution to the labelling problem $O(3) \subset U(3)$.

2. Notation and terminology

Fix a compact semisimple Lie group G, let L be its complexified Lie algebra, and let U be the universal enveloping algebra of L. Following the notation of Humphreys (1972), let H be a Cartan subalgebra of L, and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a base for the root system of H^*. The corresponding fundamental dominant weights $\{\Lambda_1, \Lambda_2, \ldots, \Lambda_l\}$ are defined from the root system via the form $\langle\,,\rangle$ given by

$$\langle \Lambda_i, \alpha_j \rangle \equiv 2(\Lambda_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$$

where $(\,,)$ denotes the dual of the Killing form on H. The positive integral linear combinations of the fundamental dominant weights form the set Λ^+ of dominant integral weights, which is in one-to-one correspondence with the set of irreducible representations (IR) of G; we shall use λ, μ, etc, to denote both highest weights and IR. The carrier space of the IR λ will be denoted $V(\lambda)$. The representation contragredient to λ, carried on the dual space $V(\lambda)^*$, will be denoted λ^*. The weights occurring in $V(\lambda)$ will be ordered, as above, from lowest to highest with $V_i(\lambda)$ the weight subspace of $V(\lambda)$ for the weight λ_i. The full lattice carries the partial order induced by the positive roots: $\lambda < \mu$ iff $\mu - \lambda$ is a sum of positive roots. It is convenient to fix a standard set of generators. For each positive root α, let h_α be the element of H satisfying $(h_\alpha, h) = \alpha(h)$, $\forall h \in H$, and fix x_α and y_α by the requirements

$$[h, x_\alpha] = \alpha(h) x_\alpha \qquad \forall h \in H$$

$$[h, y_\alpha] = \alpha(h) y_\alpha \qquad \forall h \in H$$

$$[x_\alpha, y_\alpha] = h_\alpha.$$

For $\alpha = \alpha_i \in \Delta$, write h_i, etc, for $h_{\alpha i}$. The standard set is then $\{x_i, y_i, h_i, i = 1, \ldots, l\}$. We can decompose L as

$$L = B \oplus H \oplus N$$

where B (resp N) is the nilpotent subalgebra generated by the $\{x_i\}$ (resp $\{y_i\}$). By the PBW theorem (Humphreys 1972, p92), U may be factorised as U(B)U(H)U(N) where U(B) (resp U(H), U(N)) is the subalgebra of U generated by B (resp H, N). Let δ be half the sum of the positive roots.

3. Projective resolution of the Clebsch–Gordan multiplicity

We begin by recalling some results from Gould and Edwards (1984). Our aim is to decompose the tensor product representation $V(\lambda) \otimes V(\mu)$ into irreducible subrepresentations. We fix attention on the set of irreducible subrepresentations of highest weight $\lambda_i + \mu$ where λ_i is a weight of $V(\lambda)$. Any highest weight vector of this weight must lie in the space spanned by simple product states of the form $e \otimes f$, $e \in V_k(\lambda)$, $f \in V_j(\mu)$, with $\lambda_k + \mu_j = \lambda_i + \mu$. We focus however on the subspace of the space of vectors of weight $\lambda_i + \mu$ spanned by product states of the form $e \otimes e^\mu_+$; this subspace may be written $V_i(\lambda) \otimes e^\mu_+$. The main result needed from the previous paper is one based on an earlier result of Parthasarathy *et al* (1967). It states that the multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$ is equal to the dimension of the space $V_{i,\mu}(\lambda)\subset V_i(\lambda)$, which is defined in terms of U as

$$V_{i,\mu}(\lambda)\equiv\{v\in V_i(\lambda)|x_j^{\langle\mu+\delta,\alpha_j\rangle}v=0; j=1,\ldots,l\}.$$

The multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$ is the dimension of the space of highest weight vectors in $V(\lambda)\otimes V(\mu)$ of weight $\lambda_i+\mu$. There is in fact a simple bijection between this space and $V_{i,\mu}(\lambda)$: the decomposition of any HWV of weight $\lambda_i+\mu$ into direct product states will always have a term of the form $e\otimes e_+^{\mu}$, with $e\in V_{i,\mu}(\lambda)$. Conversely, the projection of any product state of the form $e\otimes e_+^{\mu}$ with $e\in V_{i,\mu}(\lambda)$ onto the space of HWV will always be non-zero, while the same projection from a state $e\otimes e_+^{\mu}$ with e an element of the orthocomplement of $V_{i,\mu}(\lambda)$ will vanish.

The proof of this result, and specifically the definition of the space $V_{i,\mu}(\lambda)$ in terms of the group generators, is grounded in the fact that all tensor product representations of a compact semisimple Lie group are cyclic: the representation $V(\alpha)\otimes V(\beta)$ is cyclically generated by the vector $e_+^{\alpha}\otimes e_-^{\beta}$. Just as in the case of an irreducible representation (which is trivially cyclically generated from its highest weight vector) much information about the structure of the cyclic module may be obtained from a knowledge of the enveloping algebra annihilator of the cyclic vector. In the irreducible case, the well known result of Verma (1968) tells us that e_+^{α} is annihilated by the ideal

$$\sum \mathrm{U}x_i+\sum \mathrm{U}y_i^{\langle\alpha+\delta,\alpha_i\rangle}+\sum \mathrm{U}[h_i-\alpha(h_i)].$$

As shown in our previous paper, the annihilator of $e_+^{\alpha}\otimes e_-^{\beta}$ has a similar form, being

$$\sum \mathrm{U}x_i^{\langle\beta+\delta,\alpha_i\rangle}+\sum \mathrm{U}y_i^{\langle\alpha+\delta,\alpha_i\rangle}+\sum \mathrm{U}[h_i-(\alpha-\beta^*)(h_i)].$$

Projecting $e_+^{\lambda_i+\mu}\otimes e_-^{\mu^*}$ onto subrepresentations of $V(\lambda_i+\mu)\otimes V(\mu)^*$ equivalent to λ always yields an element of $V_{i,\mu}(\lambda)$, while conversely the decomposition into product states of vectors in subrepresentations of $V(\lambda_i+\mu)\otimes V(\mu^*)$ equivalent to λ will yield a non-zero component along $e_+^{\lambda_i+\mu}\otimes e_-^{\mu^*}$ for elements of $V_{i,\mu}(\lambda)$, and a zero component for elements of the orthocomplement of $V_{i,\mu}(\lambda)$. The former projections are intertwining operators for the group action, with the result that elements of $V_{i,\mu}(\lambda)$ must be anniliated by anything annihilating $e_+^{\lambda_i+\mu}\otimes e_-^{\mu^*}$. We have shown in the earlier paper that the latter condition is in fact both necessary and sufficient for a vector to be an element of $V_{i,\mu}(\lambda)$. The relationship between these properties of $V(\lambda_i+\mu)\otimes V(\mu^*)$ and its subrepresentations equivalent to λ, and the properties of subrepresentations of $V(\lambda)\otimes V(\mu)$ equivalent to $\lambda_i+\mu$ which are the subject of the present discussion, stems directly from the natural bijection between $V(\lambda_i+\mu)\otimes V(\mu^*)$ and the space of linear operators from $V(\mu)$ into $V(\lambda_i+\mu)$: the subrepresentations of $V(\lambda_i+\mu)\otimes V(\mu^*)$ equivalent to λ correspond under this bijection to the tensor operators transforming according to λ.

The space $V_{i,\mu}(\lambda)$ is vastly simpler than the space of HWV in $V(\lambda)\otimes V(\mu)$ which it labels. The set $\{x_k, y_k, h_k\}$ for each $k\in\{1,\ldots,l\}$ specifies a Lie subalgebra of L isomorphic to sl(2): denote it sl(2)$_k$. The IR of sl(2) are defined by a single highest weight integer label. The condition $y_k^{m_k+1}e=0$ with $m_k=\langle\mu,\alpha_k\rangle$ for some vector e of known weight λ_i with kth component λ_{ik} simply asserts that the spectral decomposition of e under sl(2)$_k$ contains no component lying in an sl(2)$_k$ submodule with highest weight greater than $2m_k-\lambda_{ik}$. Alternatively we can say that the kth defining condition of $V_{i,\mu}(\lambda)$ specifies that it is contained within W_k, which we define as the 'cumulative

$\mathrm{sl}(2)_k$ spectral subspace of $V_i(\lambda)$ with highest weight $2m_k-\lambda_{ik}$', i.e. the span of all vectors in $V_i(\lambda)$ which lie in $\mathrm{sl}(2)_k$ submodules of highest weight no greater than $2m_k-\lambda_{ik}$. $V_{i,\mu}(\lambda)$ is itself the intersection $W_1\cap W_2\cap\cdots\cap W_l$.

As m_k, the kth highest weight component of μ, increases, the annihilator ideal of $e_+^{\lambda_i+\mu}\otimes e_-^{\mu*}$ decreases. The cumulative spectral subspace W_k starts for low values of m_k at zero, increasing with increasing m_k (after $2m_k-\lambda_{ik}$ it reaches the value of the lowest $\mathrm{sl}(2)_k$ spectral component in $V_i(\lambda)$) until it is the whole of $V_i(\lambda)$. At this stage the component m_k has no further influence on the multiplicity which remains at its maximal value if the values of the other highest weight components of μ are held fixed. This behaviour gives rise to 'pie-shaped' regions defined by the sets of IR μ for which the multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$ for fixed λ_i is greater than some given value. Such regions have been studied in detail for U(3) (Lohe *et al* 1977) and will be discussed further in the next paper. For the vast majority of representations μ, the multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$ is either minimal (zero) or maximal (dim $V_i(\lambda)$). In the latter case any basis of $V_i(\lambda)$ will serve to label an (in general non-orthogonal) basis of the space of HWV of weight $\lambda_i+\mu$ in $V(\lambda)\otimes V(\mu)$. However we show in a subsequent paper that for large μ an orthonormal basis of $V_i(\lambda)$ gives rise asymptotically to an orthonormal resolution of the multiplicity. Additionally it is worth noting that polar decomposition of the projection acting between the space $V_{i,\mu}(\mu)$ and the space of HWV of weight $\lambda_i+\mu$ may be used to set up a natural unitary correspondence between the two spaces.

4. Computation of Wigner coefficients

The simplest labelling map from $V_{i,\mu}(\lambda)$ onto the space of HWV of weight $\lambda_i+\mu$ is projection. Let P be this projection. P can be taken as the restriction to $V_i(\lambda)\otimes e_+^{\mu}$ of the central projection in $V(\lambda)\otimes V(\mu)$ onto the space spanned by all subrepresentations of highest weight $\lambda_i+\mu$. There are two well known methods for calculating the matrix elements of central projections, which in this case yield the Wigner coefficients of the form $\langle e^i{}_j\otimes e_+^{\mu}|e_+^{\lambda_i+\mu}{}_{,j}\rangle$, where the notation for basis vectors here indicates that $\{e^i{}_j|i=1,\ldots,n;\, j=1,\ldots,\dim V_i(\lambda)\}$ is a basis for $V(\lambda)$ adapted to the weight spaces $V_i(\lambda)$ while $\{e_+^{\lambda_i+\mu}{}_{,j}\}$ for fixed i and varying j is a basis for the space of HWV in $V(\lambda)\otimes V(\mu)$ obtained by projection from the vectors $e^i{}_j$, the range of j being chosen to yield a maximal linearly independent set. The first method is integration over the group manifold, such as the Hill-Wheeler integrals used in calculating the transformation coefficients for the Elliott reduction of the $\mathrm{O}(3)\subset\mathrm{U}(3)$ problem (e.g. Moshinsky *et al* 1975). The alternative is the infinitesimal method of spectral reduction of Casimir and/or higher-order enveloping algebra invariants, such as the Gel'fand invariants of $\mathrm{U}(n)$, $\mathrm{O}(n)$ and $\mathrm{Sp}(2n)$. For the present application, the infinitesimal method is simpler in view of the fact that for a representation generated cyclically from a weight vector of weight $\lambda_i+\mu$, only the second-order Casimir invariant is needed in order to separate the IR of highest weight $\lambda_i+\mu$ from the other IR (Gould and Edwards 1984). A related use of the global projection method appears in Klimyk and Gavrilik (1979).

It is shown in the earlier paper that the central decomposition of a product state $e^i{}_j\otimes e_+^{\mu}$ yields a sum of $P(e^i{}_j\otimes e_+^{\mu})$ and components lying in IR whose highest weights are all strictly greater than $\lambda_i+\mu$ under the partial order on the weight lattice. The character of the Casimir operator C is a function on the set of dominant integral weights which is strictly monotonic with respect to that partial order. Hence the central projection for the highest weight $\lambda_i+\mu$ acting on $e^i{}_j\otimes e^\mu_+$ is implemented by

$$P=\prod_{\lambda_i<\sigma\leqslant\lambda}\left(\frac{C-\chi_{\sigma+\mu}(C)}{\chi_{\lambda_i+\mu}(C)-\chi_{\sigma+\mu}(C)}\right)$$

with $\chi_\alpha(C)$ the eigenvalue of C in the IR α; the product is only over highest weights of $V(\lambda)\otimes V(\mu)$ strictly greater than $\lambda_i+\mu$. In this expression, each term in the product is quadratic in the highest weight labels $(m_1, m_2, \ldots, m_l)$ of μ. A simpler expression for the action of P on $e^i{}_j\otimes e^\mu_+$ is obtained by replacing C by the invariant

$$\begin{aligned} A &\equiv \sum a_i\otimes a^i && \text{(with } \{a_i\}, \{a^i\} \text{ dual bases of L)}\\ &= C(\lambda\otimes\mu)-C(\lambda)\otimes I-I\otimes C(\mu) && (I\equiv\text{identity operator})\end{aligned}$$

whose eigenvalues are linear functions of $(m_1, m_2, \ldots, m_l)$. The separation property is not lost when C is replaced by A, whose eigenvalues are given explicitly by

$$\begin{aligned}\chi_{\sigma+\mu}(A) &= \chi_{\sigma+\mu}(C)-\chi_\lambda(C)-\chi_\mu(C)\\ &=(\sigma+\mu, \sigma+\mu+\delta)-(\lambda, \lambda+\delta)-(\mu, \mu+\delta)\\ &=(2\mu+\sigma+2\delta, \sigma)-(\lambda, \lambda+2\delta).\end{aligned}$$

We shall now show that the matrix elements $\langle e^i{}_j\otimes e^\mu_+|P|e^i{}_k\otimes e^\mu_+\rangle$ have the desirable property of being ratios of polynomial functions in $(m_1, m_2, \ldots, m_l)$ of degree no larger than the number of weights in $V(\lambda)$ strictly greater than λ_i. In the process program steps for the explicit computation of these matrix elements are exhibited. It is worth noting that the complete behaviour of the polynomial numerator and denominator functions is determined by their behaviour on the region of the μ lattice corresponding to maximal multiplicity of $\lambda_i+\mu$ in $\lambda\otimes\mu$. In this region the complexities of the intermediate multiplicity decompositions (neither minimal nor maximal for fixed λ_i) are avoided. This is seen from the following results (Kostant 1975, propositions 4.1 and 4.2). Define a highest weight λ to be *subordinate* to a highest weight μ if, for every weight λ_i in $V(\lambda)$, the IR $\lambda_i+\mu$ occurs in $\lambda\otimes\mu$ with maximal multiplicity dim $V_i(\lambda)$. Also let $\leqslant_0$ denote the partial order: $\mu\leqslant_0\lambda$ iff $\lambda-\mu\in\Lambda_+$, the lattice of dominant integral weights. Then

(a) for any highest weight λ there is a highest weight μ_0 such that λ is subordinate to μ for every $\mu\geqslant_0\mu_0$;

(b) a polynomial function f on the weight lattice Λ vanishes identically iff it vanishes on all μ not less than some fixed highest weight μ_0.

To see that the matrix elements $\langle e^i{}_j\otimes e^\mu_+|f(A)|e^i{}_k\otimes e^\mu_+\rangle$ are polynomials in $(m_1, m_2, \ldots, m_l)$ for any polynomial f in A, note that

$$\langle e^i{}_j\otimes e^\mu_+|A^m|e^i{}_k\otimes e^\mu_+\rangle=\sum_{i_1\ldots i_m}\langle e^i{}_j|a_{i_1}a_{i_2}\ldots a_{i_m}|\,e^i{}_k\rangle\langle e^\mu_+|a^{i_1}a^{i_2}\ldots a^{i_m}|e^\mu_+\rangle.$$

The μ dependence of this expression is entirely in the factors $\langle e^\mu_+|a^{i_1}a^{i_2}\ldots a^{i_m}|e^\mu_+\rangle$, i.e. a diagonal matrix element between highest weight states. The only non-zero terms in the sum over $i_1, \ldots, i_m$ thus arise from products $a^{i_1}a^{i_2}\ldots a^{i_m}$ that do not shift the weight. These are all elements of the algebra C(H), the centraliser in U of the Cartan subalgebra H. The whole expression (insofar as we are interested in its dependence on μ) is therefore a linear combination of diagonal matrix elements of members of C(H) taken between highest weight states of weight μ. Any element of C(H) is the sum of a polynomial in H and something which annihilates e^μ_+. (This follows by looking at monomials in C(H) and reordering terms using the commutation relations as in the proof of the PBW theorem.) A computer program to evaluate the expression would benefit by exploiting the fact that it has the structure of a linear combination of products of a factor independent of μ and a polynomial function of μ.

5. Conclusion

We have shown how the task of resolving the multiplicity in IR of weight $\lambda_i + \mu$ in the tensor product representation $\lambda \otimes \mu$ and then calculating the Wigner coefficients involving highest weight states of the resolved IR can be reduced to a study of the behaviour of $\mathrm{sl}(2)_k$ subalgebras of L on the weight subspace $V_i(\lambda) \subset V(\lambda)$. The latter problem is a vastly simpler one than the former, and we find that it gives rise to comparatively simple rational function expressions for the highest weight Wigner coefficients. Future work will be devoted to a detailed study of these polynomial functions in specific cases; an evaluation of the zeros of these polynomials and their relation to the complex structure of the Clebsch-Gordan problem in the intermediate multiplicity region will be of particular interest.

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